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The Bosonic Central Limit Theorem

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1 The Bosonic Central Limit Theorem

The aim of this article is to give an overview on recent progress of the central limit theorem for mixing quantum spin chains.

The limit theorem we discuss here is described in the language of operator algebras . (c.f. [4] , [5]) We consider the following one dimensional quantum systems acting on a Hilbert space $\mathcal{H}$. We assume that $\mathcal{A}$ is an *algebra of bounded operators on $\mathcal{H}$ the system is spatially homogenous in the sense that there exists a unitary U describing the lattice translation. Furthermore we assume the following conditions:
(i) UQU^{*} is in $\mathcal{A}$ and thus $\tau_{k}(Q)=U^{k}QU^{-k}$ gives rise an action of $\mathbf{Z}$ on $\mathcal{A}$
(ii) There exists a U invariant vector Ω which is cyclic for $\mathcal{A}$.

$$U\Omega=\Omega \quad , \quad \overline{\mathcal{A}\Omega}=\mathcal{H}$$

(iii) For any element Q of $\mathcal{A}$ corresponding to a physical observable, we assume there exists a positive integer r such that

$$[Q,\tau_{k}(Q)]=0$$

for $|k|>r$.

Consider the vector state ω (= a positive linear functional of $\mathcal{A}$) determined by

$$\omega(Q)=(\Omega,Q\Omega)$$

In this setting, we say the state ω associated with Ω is mixing if

$$\lim_{k\to\infty}\omega(R\tau_{k}(Q))=\omega(R)\omega(Q)$$

for any Q and R in $\mathcal{A}$.

Let us consider the local flucutation operator

$$Q_{<N>} = \frac{1}{\sqrt{2N+1}} \left(\sum_{|j| \leq N} (\tau_j(Q) - \omega(Q)) \right).$$

Then we introduce the (formal) global fluctuation operator $\tilde{Q}$ defined by

$$\tilde{Q} = \lim_N Q_{<N>}$$

Suppose that ω is mixing as above. Due to cyclicity and mixing property the von Neumann algebra generated by $\mathcal{A}$ is a factor i.e. with a trivial center. Then, a formal computation shows that

$$\lim_N [Q_{<N>}, R_{<N>}] = c(Q, R)1$$

where $c(Q, R)$ is a formal constant (possibly divergent). Thus the formal fluctuation operator $\tilde{Q}$ satisfies the Canonical Commutation Relations (CCR) at heuristic level.

$$\left[\tilde{Q}, \tilde{R}\right] = c(Q, R)1$$

This appearance of Boson was suggested by K.Hepp and E.Lieb in [10] and ,by Walter F.Wreszinski in [17]. Our principal interest is to justify this Bosonization of fluctuation operators. We will see that $\tilde{Q}$ exists in the sense of the classical central limit theorem and the appearance of the canonical commutation relation for $\tilde{Q}$ is justified mathematically..

A typical example of this system is the one dimensional quantum spin chain and the Hilbert spaces are obtained by the GNS construction of translationally invariant states of a UHF C^*-algebra $\mathcal{A}$. To explain the situation more precisely we introduce notations now. The UHF C^*-algebra $\mathcal{A}$ is the algebra of local physical observables and it is the infinite tensor product of the full matrix algebras. In fact we consider the algebraic tensor $\mathcal{A}_{loc}$:

$$\mathcal{A}_{loc} = \bigotimes_{\mathbf{Z}} M_d(\mathbf{C})$$

where $M_d(\mathbf{C})$ is the set of all complex matrices regarded as bounded operators on a d-dimensional Hilbert space. In what follows d is a finite integer. Each component of the tensor product above is specified with a lattice site $j \in \mathbf{Z}$. The C^*-completion of $\mathcal{A}_{loc}$ is denoted by $\mathcal{A}$.

For any integer j and any matrix Q in $M_d(\mathbf{C})$, $Q^{(j)}$ will be an observable Q located at the lattice site j. Thus, by $Q^{(j)}$ we denote the following element of $\mathcal{A}$.

$$\cdots \otimes 1 \otimes 1 \otimes \underbrace{Q}_{j} \otimes 1 \otimes 1 \otimes \cdots \in \mathcal{A}$$

Given a subset Λ of $\mathbf{Z}$, $\mathcal{A}_\Lambda$ is defined as the C^*-subalgebra of $\mathcal{A}$ generated by all $Q^{(j)}$ with $Q \in M_n(\mathbf{C})$, $j \in \Lambda$. Thus $\mathcal{A}_\Lambda$ is the set of observables localized in Λ and

$$\mathcal{A}_{loc} = \cup_{|\Lambda|<\infty} \mathcal{A}_\Lambda$$

where $|\Lambda|$ is the cardinality of Λ.

A state of $\mathcal{A}$ is a (complex valued) linear function ω on $\mathcal{A}$ satisfying the normalization condition $\omega(1) = 1$ and the positivity condition:

$$\omega(Q^*Q) \geq 0 \qquad \text{for any } Q \text{ in } \mathcal{A}\ .$$

If ω is a state of $\mathcal{A}$ the restriction of φ to $\mathcal{A}_\Lambda$ will be denoted by φ_Λ.

$$\omega_\Lambda = \varphi|_{\mathcal{A}_\Lambda}$$

Given a state ω of $\mathcal{A}$ there exist a Hilbert space $\mathcal{H}_\omega$, a unit vector Ω in $\mathcal{H}_\omega$ and a *-representation $\pi_\omega(\mathcal{A})$ of the algebra $\mathcal{A}$ acting on $\mathcal{H}_\omega$ such that $\pi_\omega(\mathcal{A})\Omega$ is dense in $\mathcal{H}_\omega$ and

$$(\Omega, \pi_\omega(Q)\Omega) = \omega(Q) \qquad \text{for any } Q \text{ in } \mathcal{A}\ .$$

The triple $\{\pi_\varphi(\mathcal{A}), \mathcal{H}_\varphi, \Omega_\varphi\}$ is unique up to unitary equivalence. It is called the GNS triple associated with the state φ and the vector Ω_φ is referred to as the GNS cyclic vector.

The translation τ_k (shift on the integer lattice $\mathbf{Z}$) is an automorphism of $\mathcal{A}$ determined by

$$\tau_k(Q^{(j)}) = Q^{(j+k)}$$

A state ω is translationally invariant if

$$\omega(\tau_k(Q)) = \omega(Q)$$

for any Q in $\mathcal{A}$ and any integer k.

First we show that the formal fluctuation operator $\tilde{Q}$ is Gaussian.

Theorem 1.1 *Let φ be a translationally invariant state . Suppose there exist positive constants $M(j)$ such that*

$$\sum_{j=1}^{\infty} jM(j) < \infty$$

and

$$|\varphi(Q_1\tau_j(Q_2)) - \varphi(Q_1)\varphi(Q_2)| \leq K\, \|Q_1\|\, \|Q_2\|\, M(j)$$

for Q_1 in $\mathcal{A}_{(-\infty,-1]}$, Q_2 in $\mathcal{A}_{[0,\infty)}$ and $j > 0$.

The central limit theorem holds for any selfadjoint strictly local observable Q in $\mathcal{A}_{loc}$ in the following sense.

$$\lim_{N\to\infty} \varphi(e^{iTQ_{<N>}}) = e^{-T^2 t(Q,Q)} \tag{1.1}$$

where

$$t(Q,R) = \lim_{N\to\infty} \varphi(Q_{<N>}R_{<N>}). \tag{1.2}$$

$\varphi(e^{iTQ_{<N>}})$ is the Fourier transform of the projection valued spectral meaure $dE_N(\lambda)$ (associated with $Q_{<N>}$) evaluated on the state φ. Thus (1.1) means that the spectrum distribution of $Q_{<N>}$ converges to a Gaussian distribution. Formally we write

$$\tilde{\varphi}(e^{iT\tilde{Q}}) = e^{-T^2 t(Q,Q)}. \tag{1.3}$$

To obtain the Canonical Commutation Relations of the algebra of fluctuation we have to "compute" the correlation of mutually non-commuting observables.

$$\tilde{\varphi}(e^{iT\tilde{Q}_1}e^{iT\tilde{Q}_2}......e^{iT\tilde{Q}_n}).$$

Theorem 1.2 *Let φ be a translationally invariant state . We assume the same uniform mixing condition as in Theorem 1. Set*

$$s(Q,R)=\sum_{k\in\mathbf{Z}}\varphi([\tau_k(Q),R]). \tag{1.4}$$

We have convergence of the following correlation functions for selfadjoint local operators $Q(k)$ $(k=1,2,...r)$ in $\mathcal{A}_{loc}$.

$$\lim_{N\to\infty}\varphi(\prod_{k=1}^{r}e^{iQ(k)<N>})=e^{-\frac{1}{2}t(\sum_k Q(k),\sum_k Q(k))}e^{-\frac{i}{2}\sum_{k<l}s(Q(k),Q(l))} \tag{1.5}$$

In the spirit of the equation (1.3) we may express (1.5) in the following manner:

$$\tilde{\varphi}(\prod_{k=1}^{r}e^{i\tilde{Q}(k)})=e^{-\frac{1}{2}t(\sum_k Q(k),\sum_k Q(k))}e^{-\frac{i}{2}\sum_{k<l}s(Q(k),Q(l))} \tag{1.6}$$

This equation (1.6) reads that the state for algebra of normal fluctuation is a quasifree state $\tilde{\varphi}$ for the Weyl form $\mathcal{W}$ of the CCR algebra. Here the algebra $\mathcal{W}$ is generated by unitaries $W(Q)=e^{i\tilde{Q}}$ for local $Q=Q*$ in $\mathcal{A}_{loc}$ and they satisfy the following relations:

$$W(Q_1)W(Q_2)=e^{-\frac{i}{2}s(Q_1,Q_2)}W(Q_1+Q_2) \tag{1.7}$$

Theorem 1.1 and 1.2 stated above were proved for the first time by D.Goderis, A.Verbeure, and P.Vets in [7] and in [8] under different mixing conditions. However they could not present any non-trivial example so it was far from obvious that the results of D.Goderis, A.Verbeure, and P.Vets have reality. As we will see in the next section, out assumption is valid for various states of one-dimensional quantum spin systems. We can prove the limit theorem for quasi local (but not strictly local) elements . Our next task is to explain the Bosonic central limit theorem for non-local operators.

Definition 1.3 *Let θ be a positive constant $0<\theta<1$. Define $\|Q\|^{(n)}$ by the following equation:*

$$\|Q\|^{(n)}=\inf\left\{\|Q-Q_n\| \mid Q\in\mathcal{A}_{[-n,n]}\right\} \tag{1.8}$$

for n positive, $n>0$ and we set

$$\|Q\|^{(0)}=\|Q\|.$$

In terms of $\|Q\|^{(n)}$ we introduce $|||Q|||_\theta$:

$$|||Q|||_\theta=\sum_{n=0,1,2,...}\|Q\|^{(n)}\theta^{-n} \tag{1.9}$$

An element Q of $\mathcal{A}$ is exponentially localized with rate θ if $|||Q|||_\theta$ is finite.
The set of all exponentially localized elements with rate θ is denoted by F_θ.
We fix an element Q_n of $\mathcal{A}_{[-n,n]}$ which attains the minimum of (1.8).

$$\|Q\|_n=\|Q-Q_n\|.$$

Theorem 1.4 *Let φ be a translationally invariant state such that*

$$|\varphi(Q_1\tau_j(Q_2)) - \varphi(Q_1)\varphi(Q_2)| \leq Ke^{-Mj}\,\|Q_1\|\,\|Q_2\| \tag{1.10}$$

for Q_1 in $\mathcal{A}_{(-\infty,-1]}$, Q_2 in $\mathcal{A}_{[0,\infty)}$ and $j > 0$. The conclusion of Theorem 1.2 is valid for for any exponentially localized selfadjoint $Q(k)$ in F_θ.

An advantage of considering the exponentially localized elements lies in the fact that we can introduce the time evolution for the algebra of normal fluctuation. Let us consider the time evolution for local observables determined by a finite range translationally invariant interaction. Our (formal) infinite volume Hamiltonian is denoted by H.

$$H = \sum_{k\in\mathbf{Z}} \tau_k(h) \tag{1.11}$$

where h is a selfadjoint local operator. Thus our time evolution α_t is generated by H in the following sense:

$$\frac{d}{dt}\alpha_t(Q) = i\,[H, \alpha_t(Q)]$$

for Q in $\mathcal{A}_{loc}$. Or we may write

$$\alpha_t(Q) = e^{itH}Qe^{-itH}.$$

Now set

$$\overline{F} = \cup_{0<\theta<1}F_\theta \quad , \quad \underline{F} = \cap_{0<\theta<1}F_\theta$$

It is known that both $\overline{F}$ and $\underline{F}$ are invariant under the time evolution α_t. Thus we can introduce the time evolution $\tilde{\alpha}_t$ for the algebra of normal fluctuation $\mathcal{W}$ via the following formula:

$$\tilde{\alpha}_t(W(Q)) = W(\alpha_t(Q)) \tag{1.12}$$

Furthermore if we can show that

$$\lim_{t\to 0} t(\alpha_t(Q) - Q, \alpha_t(Q) - Q) = 0 \tag{1.13}$$

$\tilde{\alpha}_t$ is weakly continous on the GNS space associated with $\tilde{\varphi}$. Next consider the KMS state for the time evolution generated by a finite range translationally invariant Hamiltonian. A state φ_β is a β KMS state if and only if

$$\varphi_\beta(Q_1Q_2) = \varphi_\beta(Q_2\alpha_{i\beta}(Q_1))$$

for any local Q_1 and Q_2. Physically KMS states are equilibrium states at inverse temperature β and the beta KMS state is unique for finite range translationally invariant Hamiltonians on a one-dimensional lattice .

The assumption for Theorem and the convergence (1.13) can be verified by the technique of non-commutative Ruelle operator. See [3] and [11].

Theorem 1.5 *Let φ_β be the unique β-KMS state for a finite range translationally invariant Hamiltonian of a one-dimensional quantum spin chain. The quasifree state $\tilde{\varphi}_\beta$ for the algebra of normal fluctuation is a β-KMS state for the dynamics $\tilde{\alpha}_t$ of $\mathcal{W}$ at the same inverse temperature β.*

D.Goderis, A.Verbeure, and P.Vets proved a similar KMS property under mixing assumptions. Our contribution is twofold.
(i) D.Goderis, A.Verbeure, and P.Vets could not prove the central limit theorem for non-local observables and the KMS property can not be introduced directly.
(ii) We do not verify their assumption but we can prove all the assumption stated above and Theorem 1.4 is valid without further assumption.

2 Examples

In the previous section , we discussed the Bosonic central limit theorem for KMS states briefly. Here we present other examples where we can prove assumption for the Bosonic central limit theorem.

Finitely Correlated States

The notion of finitely correlated states were first introduced by L.Accardi in [1]. More than 10 year later I.Affleck. T.Kennedy, E.Lieb and H.Tasaki discovered indepently the similar construction of states in their study of quantum spin chains. (See [2].) The construction was generalized and investigated systematically by M.Fannes , B.Nachtergaele and R.Werner in [6]. First we define a linear functional φ_Q on $\mathcal{A}_{(-\infty,0]}$ defined by

$$\varphi_Q(R) = \varphi(QR) \qquad \text{for } Q \text{ in } \mathcal{A}_{[1,\infty)} \ . \tag{2.1}$$

Following M.Fannes , B.Nachtergaele and R.Werner, we say that a translationally invariant state φ is finitely correlated if the set of linear functionals φ_Q of on $\mathcal{A}_{(-\infty,0]}$ is finite dimensional .

Proposition 2.1 *Let φ be a translationally invariant finitely correlated state. Suppose that φ is mixing in the following sense:*

$$\lim_{k\to\infty} \varphi(Q_1\tau_k(Q_2)) = \varphi(Q_1)\varphi(Q_2).$$

Then, the uniform exponential mixing (1.11) holds and the central limit theorem is valid.

Note that the exponential decay of two point correlation

$$|\varphi(Q_1\tau_k(Q_2)) - \varphi(Q_1)\varphi(Q_2)| \leq C(Q_1, Q_2)e^{-Mk}$$

is known. What matters here is the constant $C(A, B)$ on Q_1 and Q_2.

Corollary 2.2 *The central limit theorem holds for any mixing finitely correlated states and any selfadjoint element Q in $\mathcal{A}_{loc}$.*

Proposition 2.1 can be proved very easily by technique of dual transfer operator. This operator itself has been considered in the context of von Neumann algebra , however it was never applied to C^*−setting. For detail of proof of Proposition 2.1 see [13].

Quasifree States for CAR algebra

Next we take Fermion on a one dimensional lattice as another example of the Bosonic central limit theorem. The states we consider now are quasifree states of Fermions. In certain literatures of non-commutative probability theories, quasifree states are misleadingly referred to as a Fermionic analogue of gaussian measures. However if we restricted to an abelian algebra, the measure is very likely Gibbs measure for a long range classical interaction or non-Gibbsian. Thus proving the central limit theorem for abelian observables is a non-trivial matter for some quasifree states. See [14], [15], [16]. Here we concentrate on one dimensional case, though we can generalize our results to higher dimensional lattices without any difficulty.

Let $\mathcal{A}^{CAR}$ be the CAR algebra generated by Fermion creation annihilation operators c_j c_j^* . By a CAR algebra we mean a unital C^*-algebra $\mathcal{A}^{CAR}$ and $\mathcal{A}_{loc}^{CAR}$ is the dense subalgebra generated by Fermion creation annihilation operators c_j c_j^* algebraically. c_j and c_j^* satisfy the standard canonical anticommutation relations:

$$\{c_j, c_k\} = \{c_j^*, c_k^*\} = 0 \quad , \quad \{c_j, c_k^*\} = \delta_{j,k}1 \tag{2.2}$$

for any integer j and k. For $f = (f_j) \in l_2(\mathbf{Z})$ we set

$$c^*(f) = \sum_{j\in\mathbf{Z}} c_j^* f_j \quad , \quad c(f) = \sum_{j\in\mathbf{Z}} c_j f_j \tag{2.3}$$

where the sum converges in norm topology. Furthermore, let

$$B(h) = c^*(f_1) + c(f_2) \tag{2.4}$$

where $h = (f_1 \oplus f_2)$ is a vector in the test function space $\mathcal{K} = l_2(\mathbf{Z}) \oplus l_2(\mathbf{Z})$. By $\overline{f}$ we denote the complex conjugate $\overline{f} = (\overline{f}_j)$ of $f \in l_2(\mathbf{Z})$ and we introduce an antiunitary involution J on the test function space $\mathcal{K} = l_2(\mathbf{Z}) \oplus l_2(\mathbf{Z})$ determined by

$$J(f_1 \oplus f_2) = (\overline{f}_2 \oplus \overline{f}_1). \tag{2.5}$$

The (lattice) translation τ_k is introduced as follows:

$$\tau_k(c_{j+k}) = c_{j+k} \quad , \quad \tau_k(c_{j+k}^*) = c_{j+k}^*. \tag{2.6}$$

We also introduce an automorphism Θ of the CAR algebra $\mathcal{A}^{CAR}$ via the following formulae.

$$\Theta(B(h)) = -B(h)$$

for any h in $\mathcal{K}$. Then the even part $\mathcal{A}^{CAR(+)}$of $\mathcal{A}^{CAR}$ is defined as the fixed point under Θ.

$$\mathcal{A}^{CAR(+)} = \left\{ Q \in \mathcal{A}^{CAR} | \Theta(Q) = Q \right\}$$

Definition 2.3 *Let A be a positive operator on the test function space $\mathcal{K}$ satisfying*

$$0 \le A \le 1 \quad , \quad JAJ = 1 - A. \tag{2.7}$$

The quasifree state φ_A is the state of $\mathcal{A}^{CAR}$ determined by the following equations:

$$\varphi_A(B(h_1)B(h_2)...B(h_{2n+1})) = 0, \tag{2.8}$$

$$\varphi_A(B(h_1)B(h_2)...B(h_{2n})) = \sum \; sign(p) \prod_{j=1}^{n} (Jh_{p(2j-1)}, Ah_{p(2j)})_{\mathcal{K}} \tag{2.9}$$

where the sum is over all permutations p satisfying

$$p(1) < p(3) < ... < p(2n-1) \quad , \quad p(2j-1) < p(2j)$$

and $sign(p)$ is the signature of the permutation p.

Let F be the fourier transform from $\mathcal{K}$ to $\tilde{\mathcal{K}} = L_2([0, 2\pi]) \oplus L_2([0, 2\pi])$. Suppose that φ_A is translationally invariant. Then it is easy to see that the fourier transform $\overline{A} = FAF^{-1}$ is a matrix valued multiplication operator. We now state our Bosonic central limit theorem for quasifree states.

Theorem 2.4 *Let φ_A be a translationally invariant quasifree state of $\mathcal{A}^{CAR}$. The Bosonic Central Limit Theorem is valid for φ_A restricted to the local even part $\mathcal{A}^{CAR(+)} \cap \mathcal{A}^{CAR}_{loc}$ if one of the following conditions is valid:*
(a) The operator A is a projection , and all the matrix elements of $\overline{A}$ are C^∞ functions.
(b) The operator A is strictly positive , $0 < \epsilon < A < 1 - \epsilon < 1$ and all the matrix elements of $\overline{A}$ are of C^∞ class

We can also introduce the time evolution for the algebra of normal fluctuation induced by the quasifree time evolution of the CAR algebra $\mathcal{A}^{CAR}$.

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